13. Homework Assignment

Dynamical Systems II

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`http://dynamics.mi.fu-berlin.de/lectures/`

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Only attempted exercises will be discussed. The exercises are for extra credit, i.e. points you obtain will count towards your "Übungsschein", however we will not use these questions to calculate the threshold necessary to obtain the "Übungsschein".

Problem 1: Consider a parameter dependent vector field,

$$(1) \qquad \begin{array}{rcll} \dot{x} &=& f(x,\lambda), & x \in \mathbb{R}^n, \\ \dot{\lambda} &=& 0, & \lambda \in \mathbb{R}^m, \end{array}$$

in the extended phase space $\mathbb{R}^{n+m}$. Assume a trivial equilibrium, $f(0,\lambda) \equiv 0$. Thus, the linearization at the origin yields

$$A = \left(\begin{array}{c|c} D_x f(0,0) & 0 \\ \hline 0 & 0 \end{array} \right).$$

Compare the $\mathrm{ad}(A^{\mathrm{T}})$ normal form of the full system with the $\mathrm{ad}(A^{\mathrm{T}})$ normal form of

$$(2) \qquad \dot{x} = f(x,0).$$

Prove that replacing the coefficients of the normal form to (2) by suitable polynomials in λ yields the normal form to (1).

Problem 2: In class we derived the following normal form of a Takens - Bogdanov point:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a x_1^2 \\ b x_1^2 + a x_1 x_2 \end{pmatrix} + o(|x_1|^2, |x_2|^2).$$

Show, that it can be transformed to

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ y_1^2 \pm y_1 y_2 \end{pmatrix} + o(|y_1|^2, |y_2|^2).$$

Show, that the sign $\pm$ can be understood as a time reversal of the truncated normal form flow.

Problem 3: Consider the truncated vector field of Problem 2

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ y_1^2 \pm y_1 y_2 \end{pmatrix}.$$

A two-parameter unfolding of the vector field (up to time reversal) is given by the second order "pendulum"

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ y_1^2 + y_1 y_2 \end{pmatrix}.$$

Determine curves of saddle-node and Hopf bifurcations in the (λ, μ) parameter plane.

Problem 4: Consider the curves of bifurcation of Problem 3.

(i) Consider the one-dimensional center manifold of $y_1,\ y_2 = 0$ and $\lambda = 0, \mu \neq 0$. Show the existence of a heteroclinic orbit, locally, between the two equilibria bifurcating at the saddle-node bifurcation curve of Problem 3.

(ii) Consider parameters $(\mu, \lambda) \in \mathbb{R}^2$ restricted to any circle around $\lambda = 0, \mu = 0$. Can you reconcile the known Hopf bifurcation with (i)?

(iii) To discuss what happens rescale the unfolded normal form of Problem 3 by

$$\begin{aligned} y_1 &= \sigma^2 \tilde{y}_1, \\ y_2 &= \sigma^3 \tilde{y}_2, \\ \lambda &= \sigma^4 \tilde{\lambda}, \\ \mu &= \sigma^2 \tilde{\mu}, \\ t &= \sigma^{-1} \tilde{t}, \end{aligned}$$

with an arbitrary scaling parameter $\sigma > 0$. Determine the rescaled normal form in $(\tilde{y}_1, \tilde{y}_2, \tilde{\lambda}, \tilde{\mu}, \tilde{t})$ up to order one. Sketch the phase portrait of the Hamiltonian(!) vector field at $\sigma = 0$. Try to guess, or plot, the dynamics of the original vector field for all parameters outside the bifurcation curves.

References:

J. Guckenheimer and P. Holmes: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer, 2003

V.I. Arnold: Geometrical Methods in the Theory of Ordinary Differential Equations, Springer, 1988.